Lecture 2

Special Relativity – I.

Objectives:

- *To recap some basic aspects of SR*
- *To introduce important notation.*

Reading: *Schutz chapter 1; Hobson chapter 1; Rindler chapter 1.*

2.1 Introduction

The equivalence principle makes Special Relativity (SR) the starting point for GR. Familiar SR equations define much of the notation used in GR.

A defining feature of SR are the Lorentz transformations (LTs), from frame S to S' which moves at v in the +ve x-direction relative to S:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \tag{2.1}$$

$$x' = \gamma(x - vt), \tag{2.2}$$

$$y' = y, \tag{2.3}$$

$$z' = z, \tag{2.4}$$

where the Lorentz factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2.5}$$

Defining $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$, these can be re-written more

symmetrically as

$$x^{0'} = \gamma\left(x^0 - \beta x^1\right), \tag{2.6}$$
$$x^{1'} = \gamma\left(x^1 - \beta x^0\right), \tag{2.7}$$
$$x^{2'} = x^2, \tag{2.8}$$
$$x^{3'} = x^3, \tag{2.9}$$

where $\beta = v/c$, so $\gamma = (1-\beta^2)^{-1/2}$.

> NB. The indices here are written as superscripts; do not confuse with exponents! The dashes for the new frame are applied to the indices following Schutz.

More succinctly we have

$$x^{\alpha'} = \sum_{\beta=0}^{\beta=3} \Lambda^{\alpha'}{}_{\beta}\, x^{\beta},$$

for $\alpha' = 0$, 1, 2 or 3, where the coefficients $\Lambda^{\alpha'}{}_{\beta}$ represent the LT taking us from frame S to S'. Can write as a matrix:

$$\Lambda^{\alpha'}{}_{\beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.10}$$

with α' the row index and β the column index. Better still, using Einstein's summation convention write simply:

$$\boxed{x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\, x^{\beta}.} \tag{2.11}$$

> NB. The summation convention here is special: summation implied only when the repeated index appears <u>once up, once down</u>. The LT coefficients $\Lambda^{\alpha'}{}_{\beta}$ have been carefully written with a subscript to allow this. This helps keep track of indices by making some expressions, e.g. $\Lambda^{\alpha'}{}_{\beta}\, x^{\alpha'}$, invalid.

LT from S' to S is easily seen to be

$$\boxed{x^{\alpha} = \Lambda^{\alpha}{}_{\beta'}\, x^{\beta'},} \tag{2.12}$$

where

$$\Lambda^{\alpha}{}_{\beta'} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.13}$$

It is easily shown that

Prove this.

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Defining the Kronecker delta $\delta^\alpha_\beta = 1$ if $\alpha = \beta$, $= 0$ otherwise, this equation can be written:

$$\boxed{\Lambda^\alpha{}_{\gamma'}\Lambda^{\gamma'}{}_\beta = \delta^\alpha_\beta.} \tag{2.14}$$

Guarantees that after LTs from S to S' then back to S we get x^α again since

$$\Lambda^\alpha{}_{\gamma'}\Lambda^{\gamma'}{}_\beta \, x^\beta = \delta^\alpha_\beta x^\beta = x^\alpha.$$

Prove each step of this equation.

Note the use of dummy index γ' to avoid a clash with α or β.

2.2 Nature of LTs

In SR the coefficients of the LT are constant and thus

$$x^{\alpha'} = \Lambda^{\alpha'}{}_\beta \, x^\beta,$$

is a linear transform, mathematically very similar to spatial rotations such as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $c = \cos\theta$, $s = \sin\theta$, $c^2 + s^2 = 1$. A defining feature of rotations is that <u>lengths are preserved</u>, i.e.

$$l^2 = (x')^2 + (y')^2 = x^2 + y^2.$$

Q: What general linear transform

$$\begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \gamma x + \delta y, \end{aligned}$$

where α, β, γ and δ are constants, preserves lengths?

Since

$$(x')^2 + (y')^2 = \left(\alpha^2 + \gamma^2\right) x^2 + 2\left(\alpha\beta + \gamma\delta\right) xy + \left(\beta^2 + \delta^2\right) y^2,$$

then

$$\begin{aligned}\alpha^2 + \gamma^2 &= 1,\\ \alpha\beta + \gamma\delta &= 0,\\ \beta^2 + \delta^2 &= 1.\end{aligned}$$

These are satisfied by $\gamma = -\beta$ and $\delta = \alpha$, so

$$\begin{aligned}x' &= \alpha x + \beta y,\\ y' &= -\beta x + \alpha y,\end{aligned}$$

with $\alpha^2 + \beta^2 = 1$.

Thus the requirement to preserve length <u>defines</u> the linear transform representing rotations.

The "interval"

$$s^2 = (ct)^2 - x^2 - y^2 - z^2,$$

plays the same role in SR.